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On the Flow of a Stratified Fluid over a Barrier

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SUMMARY

The flow of a stratified fluid over a barrier is considered. The initial-value problem is solved, and it is shown that there is no disturbance far upstream of the barrier when $t \to \infty$. The stationary solution which is obtained by letting $t \to \infty$, is given in a simple closed form for a barrier of an arbitrary shape.

1. Introduction

In this paper we are concerned with the flow of a stratified fluid over a barrier (Long's model). A barrier is placed in a stratified fluid in equilibrium. At a given point of time the barrier is suddenly set into motion, and is given a constant velocity. The resulting flow is considered relative to a system fixed to the barrier. The shape of the barrier is quite arbitrary; it is not necessarily an isolated one. If we let $t \to \infty$ in our solution, we obtain a stationary solution which satisfies the equation of Long ([1], p. 83) and with no waves far upstream irrespective of the value of the Froude number, F.

It is known that the solution of the corresponding stationary problem as considered by Long [2], is indeterminate* when $F < \pi^{-1}$, and a condition that there shall be no disturbance far upstream of the barrier has to be put on the solution. This difficulty is avoided when treating the problem as an initial-value problem; we then obtain a uniquely determined solution with no waves far upstream. Experimentally one has observed the phenomenon of blocking (see [2]). In our opinion this must be a non-linear phenomenon. At least this paper, which deals with the linear theory, supports Long's assumption of no disturbance far upstream of the barrier. This is also in agreement with the results obtained in [3] and [4].

2. Formulation of the Problem

The problem to be studied is assumed to be two-dimensional, and is considered in an x–z-plane. The fluid which is stratified, incompressible and inviscid, is confined between two rigid boundaries; the bottom with the shape given by the eq. $z=\eta(x)$, and the horizontal plane at $z=d$. The fluid is of infinite extend in the x-direction.

The problem is treated as an initial-value problem. For $t<0$ the fluid is in equilibrium ($\boldsymbol{v}' \equiv 0$), and the density field is given by $\rho=\rho_0-\beta z$, (ρ_0 and $\beta>0$ are constants). At $t=0$ the barrier is set into motion instantaneously. It is given a constant velocity $-U\boldsymbol{i}$, where $\boldsymbol{i}$ is the unit vector in x-direction. During the small time interval which it takes the barrier to attain its velocity there is no generation of vorticity, (see [5], p. 471). Therefore, since the vorticity is zero before the barrier is set into motion, $\nabla \times \boldsymbol{v}'=0$ at $t=0^+$, i.e. immediately after the barrier has attained its velocity. Also the displacement of the fluid particles is negligible during the small time interval which it takes the barrier to attain its velocity, i.e. the density field at $t=0^+$ is the same as immediately before the barrier is set into motion, and we must have that:

$$\rho_0 \frac{D\boldsymbol{v}'}{Dt} = -\nabla p-(\rho_0-\beta z)g\boldsymbol{k} \text{ at } t=0^+ ,$$

where $\boldsymbol{k}$ is the unit vector in the z-direction. It is assumed that the density stratification is so

* Long has remarked that this indeterminancy can be removed by introducing a fictitious viscosity.

small that the Boussinesq approximation can be applied; therefore ρ_0 in the inertial term. From this eq. we get:

$$\nabla \times \rho_0 \frac{D\boldsymbol{v}'}{Dt} = 0 \text{ at } t=0^+ .$$

From now on and in the rest of this paper the flow is referred to a system which is fixed to the barrier. In this system there is a basic motion of the fluid given by:

$$\boldsymbol{v}_b = U\boldsymbol{i}\,, \qquad \rho_b = \rho_0 - \beta z \tag{2.1}$$

The equations governing the motion are the hydrodynamic equations for an incompressible and inviscid fluid. These equations are linearized, assuming the disturbances in the velocity- and density field to be small compared with $\boldsymbol{v}_b$ and ρ_b respectively, and the stream function $\Psi(x, z, t)$ is introduced. $\Psi(x, z, t)$ is defined by

$$\boldsymbol{v}_1 = -\nabla \times \Psi(x, z, t)\boldsymbol{j}\,, \tag{2.2}$$

where $\boldsymbol{j}$ is the unit-vector in the y-direction and $\boldsymbol{v}_1$ denotes the disturbance in the velocity field.

Because the Boussinesq approximation is applied, we obtain the following equation for $\Psi(x, z, t)$:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \nabla^2 \Psi + \hat{g}\frac{\partial^2 \Psi}{\partial x^2} = 0\,, \qquad \hat{g} = \frac{g\beta}{\rho_0}. \tag{2.3}$$

The boundary conditions are that the velocity component normal to the boundaries must be equal to zero. Linearizing the boundary conditions, we obtain:

$$\left.\begin{aligned} v_{1z} &= -\frac{\partial \Psi}{\partial x} = U\frac{\partial \eta(x)}{\partial x} \text{ at } z=0\,, \\ v_{1z} &= 0 \text{ at } z=d\,, \end{aligned}\right\} \tag{2.4}$$

where $z=\eta(x)$ represents the shape of the barrier. $\eta(x)$ is supposed to be a continuous function of x, and $\eta(x)\to 0$ when $|x|\to\infty$ so that $\int_{-\infty}^{\infty}|\eta(x)|\,dx$ exists.

The initial conditions follow easily from the discussion at the beginning of this section. They are:

$$\left.\begin{aligned} &\nabla^2\Psi = 0\,, \\ &\frac{D}{Dt}\nabla^2\Psi = 0, \text{ which in linearized form is:} \\ &\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\Psi = 0\,. \end{aligned}\right\} \text{at } t=0^+ . \tag{2.5}$$

3. The Solution of the Problem*

In (2.3) we use the Fourier and the Laplace transformation to obtain

$$(p+ikU)^2\left(\frac{d^2}{dz^2} - k^2\right)\bar{\psi}(k, z, p) - k^2\hat{g}\bar{\psi}(k, z, p) = 0\,, \tag{3.1}$$

where

$$\psi(k, z, t) = F.T.\{\Psi(x, z, t)\} = \int_{-\infty}^{\infty} \Psi(x, z, t)\,e^{-ikx}\,dx\,.$$

$$\bar{\psi}(k, z, p) = \int_0^{\infty} \psi(k, z, t)\,e^{-pt}\,dt\,.$$

* After the submission of this paper there has appeared a paper by J. W. Miles: Transient motion of a dipole in a rotating flow, *J. Fluid Mech.*, 39, (1969) 433, where that problem is solved in an analogous manner.

Also from (2.4) we obtain

$$\left.\begin{aligned}\bar{\psi} &= \frac{F(k)}{p} \text{ at } z=0, \text{ where } F(k)=F.T.\{-U\eta(x)\}\\ \bar{\psi} &= 0 \text{ at } z=d\,.\end{aligned}\right\} \tag{3.2}$$

From (3.1) we obtain

$$\frac{d^2\bar{\psi}}{dz^2} + \left(-k^2 + \frac{\hat{g}}{(U-\zeta)^2}\right)\bar{\psi} = 0, \text{ where } \zeta = \frac{ip}{k}\,. \tag{3.3}$$

A solution of this equation, which satisfies the boundary conditions (3.2), is

$$\bar{\psi} = \frac{F(k)}{p}\,\frac{\sin\omega(d-z)}{\sin\omega d}\,, \text{ where } \omega^2 = -k^2 + \frac{\hat{g}}{(U-\zeta)^2}\,. \tag{3.4}$$

To find $\psi(k, z, t)$ we have to perform the inversion, i.e.

$$\psi(k, z, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{\psi}\, e^{tp}\, dp = \frac{k\,\mathrm{sgn}\,k}{2\pi}\int_{i\zeta_0-\infty}^{i\zeta_0+\infty} \bar{\psi}\, e^{-ik\zeta t} d\zeta\,. \tag{3.5}$$

The line of integration is above or below the real axis in the complex ζ-plane according to whether $k>0$ or $k<0$.

The singularities of $\bar{\psi}$ are:

(1) Poles at $\zeta=0$, $\zeta_n^- = U - \dfrac{\hat{g}^{\frac{1}{2}}}{\left\{\left(\frac{n\pi}{d}\right)^2 + k^2\right\}^{\frac{1}{2}}}$,

$$\zeta_n^+ = U + \frac{\hat{g}^{\frac{1}{2}}}{\left\{\left(\frac{n\pi}{d}\right)^2 + k^2\right\}^{\frac{1}{2}}}, \quad n=1, 2, \dots\,.$$

(2) A non-isolated essential singularity at $\zeta=U$.

$\omega=\{-k^2+\hat{g}/(U-\zeta)^2\}^{\frac{1}{2}}$ is a two-valued function of ζ with branch-points at $\zeta=U\pm\hat{g}^{\frac{1}{2}}/k$. However, we find that these points are not branch-points for $\bar{\psi}$. To evaluate the integral in (3.5) we perform an integration in the complex ζ-plane. We obtain

$$\left.\begin{aligned}\psi(k, z, t) = F(k)\frac{\sin\omega_0(d-z)}{\sin\omega_0 d} &- F(k)\sum_{n=1}^{\infty}\frac{a_n}{\zeta_n^-}\sin\frac{n\pi z}{d}\,e^{-ik\zeta_n^- t}\\ &+ F(k)\sum_{n=1}^{\infty}\frac{a_n}{\zeta_n^+}\sin\frac{n\pi z}{d}\,e^{-ik\zeta_n^+ t}\,,\end{aligned}\right\} \tag{3.6}$$

where

$$\omega_0 = \left\{\frac{\hat{g}}{U^2} - k^2\right\}^{\frac{1}{2}}, \qquad a_n = \frac{n\pi}{d^2}\,\frac{\hat{g}^{\frac{1}{2}}}{\left\{\left(\frac{n\pi}{d}\right)^2 + k^2\right\}^{\frac{3}{2}}}\,.$$

The solution of the problem is now found by performing the inversion, i.e.

$$\Psi(x, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(k, z, t)\, e^{ikx} dk\,. \tag{3.7}$$

When $\psi(k, z, t)$ has poles on the real k-axis, the principal value of the integral is to be taken.

4. Investigation of the Time-Dependent Part of the Solution

In this section we will study the asymptotic behaviour of the solution (3.7) for x fixed and $t\to\infty$.

We introduce (3.6) into (3.7) and find that we have to investigate how the following types of integrals behave asymptotically:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\frac{a_n}{\zeta_n^-}\mathrm{e}^{ik(x-\zeta_n^- t)}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\frac{a_n}{\zeta_n^-}\mathrm{e}^{ikx}\,\mathrm{e}^{tf_1(k)}dk \tag{4.1}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\frac{a_n}{\zeta_n^+}\mathrm{e}^{ik(x-\zeta_n^+ t)}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\frac{a_n}{\zeta_n^+}\mathrm{e}^{ikx}\,\mathrm{e}^{tf_2(k)}dk\,, \tag{4.2}$$

where we have written

$$f_1(k) = -ik\left\{U - \frac{\hat{g}^{\frac{1}{2}}}{\left\{\left(\frac{n\pi}{d}\right)^2 + k^2\right\}^{\frac{1}{2}}}\right\},$$

$$f_2(k) = -ik\left\{U + \frac{\hat{g}^{\frac{1}{2}}}{\left\{\left(\frac{n\pi}{d}\right)^2 + k^2\right\}^{\frac{1}{2}}}\right\}.$$

We choose that branch of the two-valued function $\{(n\pi/d)^2+k^2\}^{\frac{1}{2}}$ which has the real part $\geqq 0$. Let

$$N\pi < F^{-1} < (N+1)\pi\,,$$

where N is an integer, and $F = U/\hat{g}^{\frac{1}{2}}d$ is the Froude number.

We find that

$$\zeta_n^- = 0 \quad \text{for} \quad k = \pm k_n = \pm\left\{\frac{\hat{g}}{U^2} - \left(\frac{n\pi}{d}\right)^2\right\}^{\frac{1}{2}}, \tag{4.3}$$

where k_n is real when $n \leqq N$ and purely imaginary when $n > N$.

To study the asymptotic behaviour of the integrals (4.1) and (4.2) we will use the method of steepest descents [6].

Let us start with the integral in (4.1). The saddlepoints are given by the equation

$$f_1'(k) = 0\,,$$

where the prime indicates differentiation with respect to k. The saddlepoints are:

$$k = \pm k_{ns} = \pm\left\{-\left(\frac{n\pi}{d}\right)^2 + \left(\frac{\left(\frac{n\pi}{d}\right)^2 \hat{g}^{\frac{1}{2}}}{U}\right)^{\frac{2}{3}}\right\}^{\frac{1}{2}},$$

which are real when $n \leqq N$ and purely imaginary when $n > N$. Also $|k_{ns}| < |k_n|$.

Applying the method of steepest descents to the integrals in (4.1) and (4.2), we deform the original path of integration (the real k-axis) to a path (denoted by Γ) which consists of a set of lines along which Re $f_{1(2)} = \text{const.} < 0$ or Im $f_{1(2)} = \text{const.}$, going through the appropriate saddlepoint(s). In the following it is assumed that $F(k)$ is analytic on and in the region between the real k-axis and Γ. We obtain:

1. *When* $n \leqq N$,

$$\frac{1}{2\pi}P\int_{-\infty}^{\infty}(\ldots)\mathrm{e}^{tf_1(k)}dk = \frac{1}{2\pi}\int_{\Gamma}(\ldots)\mathrm{e}^{tf_1(k)}dk - \\ - \frac{i}{2}\left[F(k_n)R_{k_n}\mathrm{e}^{ik_n x} + F(-k_n)R_{-k_n}\mathrm{e}^{-ik_n x}\right], \tag{4.4}$$

where $R_{\pm k_n}$ are the residues of the function (a_n/ζ_n^-) at its poles $k = \pm k_n$. In this case Γ goes through the saddlepoints $\pm k_{ns}$, which are real. Now

$$\frac{1}{2\pi}\int_{\Gamma}(\ldots)e^{tf_1(k)}dk = 0(t^{-\frac{1}{2}}) \text{ when } t\to\infty . \tag{4.5}$$

The main contribution to the asymptotic expression for the integral in (4.5) comes from the regions around the saddlepoints. From (4.4) and (4.5) we obtain:

$$\frac{1}{2\pi}P\int_{-\infty}^{\infty}(\ldots)e^{tf_1(k)}dk \to -\frac{i}{2}[F(k_n)R_{k_n}e^{ik_nx}+F(-k_n)R_{-k_n}e^{-ik_nx} \tag{4.6}$$

when $t\to\infty$.

2. *When* $n>N$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(\ldots)e^{tf_1(k)}dk = \frac{1}{2\pi}\int_{\Gamma}(\ldots)e^{tf_1(k)}dk = 0(t^{-\frac{1}{2}}\exp\{t \operatorname{Re} f_1(-k_{ns})\}) \text{ when } t\to\infty .$$

Γ goes through the saddlepoint $-k_{ns}$, which is purely imaginary. Re $f_1(-k_{ns})<0$ and consequently the integral in (4.1) with $n>N$ tends to zero when $t\to\infty$.

3. It remains to give the asymptotic behaviour of the integral in (4.2). The saddlepoints are given by the equation $f'_2(k)=0$, which is satisfied by:

$$k=\pm A^{\frac{1}{2}}\pm iB^{\frac{1}{2}}, \text{ where}$$

$$A=-\frac{1}{2}\left\{\left(\frac{n\pi}{d}\right)^2+\frac{C}{2}\right\}+\frac{1}{2}\left(\left\{\left(\frac{n\pi}{d}\right)^2+\frac{C}{2}\right\}^2+\frac{3}{4}C^2\right)^{\frac{1}{2}},$$

$$B=\frac{1}{2}\left\{\left(\frac{n\pi}{d}\right)^2+\frac{C}{2}\right\}+\frac{1}{2}\left(\left\{\left(\frac{n\pi}{d}\right)^2+\frac{C}{2}\right\}^2+\frac{3}{4}C^2\right)^{\frac{1}{2}},$$

$$C=\left(\frac{\hat{g}^{\frac{1}{2}}\left(\frac{n\pi}{d}\right)^2}{U}\right)^{\frac{2}{3}}.$$

In this case Γ goes through the saddlepoints $\pm A^{\frac{1}{2}}-iB^{\frac{1}{2}}$, and we obtain:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(\ldots)e^{tf_2(k)}dk = \frac{1}{2\pi}\int_{\Gamma}(\ldots)e^{tf_2(k)}dk =$$
$$=0(t^{-\frac{1}{2}}\exp\{t \operatorname{Re} f_2(A^{\frac{1}{2}}-iB^{\frac{1}{2}})\}) \text{ when } t\to\infty .$$

$$(\operatorname{Re} f_2(A^{\frac{1}{2}}-iB^{\frac{1}{2}})=\operatorname{Re} f_2(-A^{\frac{1}{2}}-iB^{\frac{1}{2}})) .$$

We find that Re $f_2(A^{\frac{1}{2}}-iB^{\frac{1}{2}})<0$, and therefore the integral in (4.2) tends to zero when $t\to\infty$.

In the foregoing we have assumed that $F(k)$ is analytic on and in the region between the real k-axis and Γ. For a given barrier it has to be investigated whether this assumption is satisfied or not. When the barrier is that of Long, i.e. when

$$\eta(x)=\frac{a}{2}\left(1+\cos\frac{\pi x}{b}\right)(H(x+b)-H(x-b))^{\star} , \tag{4.7}$$

it is found that $F(k)$ is analytic in the whole k-plane. Consequently the results above are valid for this barrier. We arrive at the same conclusion when the barrier is a general isolated one.

When the barrier is that of Queney, ([1], p. 68), $F(k)$ is not analytic in the whole k-plane, and the results above cannot be applied directly. However, we can split the integral $\int_{-\infty}^{\infty}$ into the two integrals $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$, and in these two integrals $F(k)$ is analytic, and we can find the asymptotic behaviour of them separately. It is to be noted that there will be an asymptotic contribution of order t^{-1} from the region near the origin in both of these integrals. However, this contribution tends to zero when $t\to\infty$, and consequently the main result above, which is expressed by (4.6), is still valid.

⋆ $H(x)$ is the Heaviside unit function.

5. The General Stationary Solution

Taking into account (3.6), (3.7) and the results from section 4, we find that the stationary solution is given by

$$\left.\begin{aligned}\Psi_s(x,z) &= \frac{1}{2\pi} P\int_{-\infty}^{\infty} F(k)\,\frac{\sin\omega_0(d-z)}{\sin\omega_0 d}\,e^{ikx}\,dk + \\ &+ \frac{i}{2}\sum_{n=1}^{N}\left[F(k_n)R_{k_n}e^{ik_n x} + F(-k_n)R_{-k_n}e^{-ik_n x}\right]\sin\frac{n\pi}{d}z\,.\end{aligned}\right\}\tag{5.1}$$

When $\sin\omega_0 d$ has zeros for real k, the principal value of the integral is to be taken, as was pointed out in section 3.

The poles of the integrand in (5.1) are given by the zeros of $\sin\omega_0 d$, which are found to be:

$$k = \pm k_n = \pm\left\{\frac{\hat{g}}{U^2} - \left(\frac{n\pi}{d}\right)^2\right\}^{\frac{1}{2}} \qquad n = 1, 2, \ldots\,.$$

We observe that this expression is the same as the expression (4.3).

ω_0 has branch-points at $k = \pm\hat{g}^{\frac{1}{2}}/U$, but we find that these points are not branch-points for the integrand in (5.1). Let us define:

$$\left.\begin{aligned}g(v) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)\,e^{ikv}\,dk\,,\\ &\text{where}\\ G(k) &= \left\{\frac{\sin\omega_0(d-z)}{\sin\omega_0 d} - \sum_{n=1}^{N}\left(\frac{R_{k_n}}{k-k_n} + \frac{R_{-k_n}}{k+k_n}\right)\sin\frac{n\pi}{d}z\right\}.\end{aligned}\right\}\tag{5.2}$$

Introducing (5.2) into (5.1) and applying the well-known Parseval's theorem from the theory of Fourier analysis ([7], p. 16), we obtain:

$$\Psi_s(x,z) = -U\left[\int_{-\infty}^{\infty}\eta(x-v)\,g(v)\,dv - 2\sum_{n=1}^{N}\frac{n\pi}{k_n d^2}\sin\frac{n\pi}{d}z\int_{-\infty}^{\infty}\eta(x-v)\sin k_n v\,H(v)\,dv\right],$$

where we have used that $R_{\pm k_n} = \pm n\pi/k_n d^2$. (5.3)

From (5.2) we obtain, performing the integration in the complex k-plane and using Cauchy's residue theorem:

$$g(v) = \left\{\begin{array}{ll}\dfrac{1}{d^2}\displaystyle\sum_{n=N+1}^{\infty}\frac{n\pi}{\kappa_n}\sin\frac{n\pi}{d}z\,e^{-\kappa_n|v|} & \text{for}\quad z\neq 0\,,\\ \delta(v) & \text{for}\quad z=0\,,\end{array}\right\}\tag{5.4}$$

where $\delta(v)$ is the Dirac δ-function, and

$$\kappa_n = \left\{\left(\frac{n\pi}{d}\right)^2 - \frac{\hat{g}}{U^2}\right\}^{\frac{1}{2}}.$$

When $z\to 0$ in the first expression in (5.4), it tends to $\delta(v)$, see Appendix. (In Appendix it is shown that $\Psi_s(x,z)\to -U\eta(x)$ when $z\to 0$, and $\Psi_s(x,z)\to 0$ when $z\to d$, i.e. $\Psi_s(x,z)$ satisfies the boundary conditions).

$\Psi_s(x,z)$ given by (5.3) gives the stationary flow over a barrier of an arbitrary shape $\eta(x)$, satisfying the condition put on it in section 2. It is not necessary to assume that the barrier is an isolated one, which is the case studied by Long.

The total stream function is:

$$\Psi_T(x,z) = Uz + \Psi_s(x,z)\,.$$

It is easily verified that $\Psi_T(x,z)$ satisfies the equation used by Long to find a solution to the stationary problem above, ([1], p.83)*.

* It is to be noted that the quantities that enter into the equation of Long are dimensionless.

Let us consider the wave-solution, which is given by the last term in (5.3). We see that:

$$\int_{-\infty}^{\infty} \eta(x-v)\sin k_n v H(v)\,dv = \int_{-\infty}^{x} \eta(v)\sin k_n(x-v)\,dv \to 0$$

when $x \to -\infty$.

Therefore, in the general case there is no wave-solution far upstream, and we conclude that the upstream condition assumed by Long is valid at least in the linear theory. (In the special case when the barrier is an isolated one, i.e. $\eta(x) = h(x)(H(x+b) - H(x-b))$, we find that:

$$\int_{-\infty}^{\infty} \eta(x-v)\sin k_n v H(v)\,dv = 0 \quad \text{for} \quad x \leqq -b\,,$$

which shows that there is no wave-solution upstream, i.e. for $x \leqq -b$).

To compare our solution with Long's solution, we introduced into (5.3) the expression for the barrier given by (4.7) and carried out the integration, (which is possible in this simple case). We obtained the same $\Psi_s(x, z)$ as Long, except for the sign. This discrepancy in sign is due to the definition of Ψ (cf. 2.2), and is obviously immaterial. We obtain the same stationary flow pattern as Long does.

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Appendix

We will show that $\Psi_s(x, z)$ tends to $-U\eta(x)$ when $z \to 0$, and to zero when $z \to d$. $\Psi_s(x, z)$ is given by (5.3).

1. Let us first prove that $\Psi_s(x, z) \to -U\eta(x)$ when $z \to 0$. It is easy to show that:

$$\left.\begin{aligned} &\frac{1}{d^2}\sum_{n=N+1}^{\infty}\frac{n\pi}{\kappa_n}\sin\frac{n\pi}{d}z\,e^{-\kappa_n|v|} \to \frac{1}{d}\sum_{n=1}^{\infty}\sin\frac{n\pi}{d}z\exp\left(-\frac{n\pi}{d}|v|\right) = \\ &= \frac{1}{d}\,\frac{\sin\frac{\pi}{d}z\left[\exp\left(-\frac{\pi}{d}v\right)\right]}{1+\left[\exp\left(-\frac{2\pi}{d}v\right)\right]-2\left[\exp\left(-\frac{\pi}{d}v\right)\right]\cos\frac{\pi}{d}z}\quad(-\infty<v<\infty),\ \text{when } z\to 0.\end{aligned}\right\}\quad \text{(A1)}$$

Also

$$\sum_{n=1}^{N}\frac{n\pi}{k_n d^2}\sin\frac{n\pi}{d}z\int_{-\infty}^{\infty}\eta(x-v)\sin k_n v\,H(v)\,dv \to 0 \text{ when } z\to 0\,. \quad \text{(A2)}$$

Taking into account (A1) and (A2), we find that

$$\left.\begin{aligned} &\Psi_s(x,z) \underset{z\to 0}{\to} -\frac{U}{d}\int_{-\infty}^{\infty}\eta(x-v)\frac{\sin\frac{\pi}{d}z\left[\exp\left(-\frac{\pi}{d}v\right)\right]}{1+\left[\exp\left(-\frac{2\pi}{d}v\right)\right]-2\left[\exp\left(-\frac{\pi}{d}v\right)\right]\cos\frac{\pi}{d}z}\,dv = \\ &-\frac{U}{2\pi i}\int_{0}^{\infty}\eta\left(x+\frac{d}{\pi}\ln v\right)\left\{\frac{1}{v-v_1}-\frac{1}{v-v_2}\right\}dv\,,\end{aligned}\right\}\quad \text{(A3)}$$

where $v_1 = \exp\left(\frac{i\pi}{d}z\right)$, $v_2 = \exp\left(-\frac{i\pi}{d}z\right)$.

But

$$\left.\begin{aligned}&\int_0^\infty \eta\left(x+\frac{d}{\pi}\ln v\right)\frac{dv}{v-v_1}\to \pi i\eta(x)+P\int_0^\infty \eta\left(x+\frac{d}{\pi}\ln v\right)\frac{dv}{v-1},\\&\int_0^\infty \eta\left(x+\frac{d}{\pi}\ln v\right)\frac{dv}{v-v_2}\to -\pi i\eta(x)+P\int_0^\infty \eta\left(x+\frac{d}{\pi}\ln v\right)\frac{dv}{v-1},\\&\qquad\qquad\qquad\qquad\qquad\qquad\text{when } z\to 0\,,\end{aligned}\right\}\tag{A4}$$

([8], p. 42)*.

Due to (A3) and (A4) we have that:

$$\Psi_s(x,z)\to -U\eta(x)\quad\text{when}\quad z\to 0\,.$$

2. To prove that $\Psi_s(x,z)\to 0$ when $z\to d$, we introduce $z=d-z'$ into (5.3) and let $z'\to 0$. We find that:

$$\frac{1}{d^2}\sum_{n=N+1}^{\infty}\frac{n\pi}{\kappa_n}\sin\frac{n\pi z}{d}\,e^{-\kappa_n|v|}=\frac{1}{d^2}\sum_{n=N+1}^{\infty}(-1)^{n+1}\frac{n\pi}{\kappa_n}\sin\frac{n\pi z'}{d}\,e^{-\kappa_n|v|}\to$$

$$\frac{1}{d}\,\frac{\sin\frac{\pi}{d}z'\exp\left(-\frac{\pi}{d}v\right)}{1+\left[\exp\left(-2\frac{\pi}{d}v\right)\right]+2\left[\exp\left(-\frac{\pi}{d}v\right)\right]\cos\frac{\pi}{d}z'}\quad(-\infty<v<\infty),\ \text{when}\ z'\to 0\,.$$

The last expression tends to zero when $z'\to 0$ for all values of v, and consequently $\Psi_s(x,z)\to 0$ when $z\to d$.

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* When we have an isolated barrier, the integrals on the right side of (A4) exist in the ordinary sense when $x\leqq -b$ and $x\geqq b$.